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# Superconformal algebras and mock theta functions

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## Abstract

It is known that characters of BPS representations of extended superconformal algebras do not have good modular properties due to extra singular vectors coming from the BPS condition. In order to improve their modular properties we apply the method of Zwegers which has recently been developed to analyze modular properties of mock theta functions. We consider the case of the  $\mathcal{N} = 4$  superconformal algebra at general levels and obtain the decomposition of characters of BPS representations into a sum of simple Jacobi forms and an infinite series of non-BPS representations. We apply our method to study elliptic genera of hyper-Kähler manifolds in higher dimensions. In particular, we determine the elliptic genera in the case of complex four dimensions of the Hilbert scheme of points on  $K3$  surfaces  $K^{[2]}$  and complex tori  $A^{[3]}$ .

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## 1. Introduction

Superconformal field theory (SCFT) in two dimensions provides a fundamental tool in describing string theory compactified on some target manifold  $X$ . One considers SCFT with an extended world sheet  $\mathcal{N} = 2$  or 4 supersymmetry depending on whether the manifold  $X$  is Kähler or hyper-Kähler. It is well known that the special feature of SCFT with extended SUSY is the existence of short or BPS representations in addition to the long or non-BPS representations. BPS representations appear at special values of conformal dimensions  $h$  and describe massless states in compactified string theory which encode the geometrical information of target manifolds. On the other hand, non-BPS representations appear at continuous values of  $h$  and correspond to massive excitations in string theory. As we shall see, the characters of BPS representations, in general, do not have a simple transformation law under modular transformations: this is due to the denominator factors which come from the BPS condition that the supercharges annihilate BPS states.

In this paper, we would like to propose a method which replaces the characters of BPS representations by some Jacobi forms, i.e. forms with good modular properties obtained by adding an infinite series of characters of non-BPS representations. This construction is based on an analogy with Zwegers' treatment of mock theta functions, where mock theta functions are replaced by (real analytic) Jacobi forms by the addition of Eichler integral of suitable 'shadows' of mock theta functions.

In the following, we restrict our discussions to the case of the  $\mathcal{N} = 4$  SCFT; however, very similar structures emerge also in the  $\mathcal{N} = 2$  case.

The study of the  $\mathcal{N} = 4$  superconformal field theory was initiated some time ago in [8–10]. The  $\mathcal{N} = 4$  superconformal algebra (SCA) contains, besides the energy–momentum tensor, four supercurrents and a triplet of currents which constitute the affine Lie algebra  $SU(2)_k$ . There exist two types of representations, i.e. BPS and non-BPS representation. We denote their character formulae as massless and massive characters, respectively. Complexity of modular properties of massless characters has been noted in [5, 6] and the level- $k = 1$  case has been considered in detail in connection with the geometry of the  $K3$  surface.

The difficulty of the massless character comes from the fact that it has the form of Lerch sum as in (2.3). The Lerch sum has long been studied in analytic number theory in connection with the so-called mock theta functions which first appeared in Ramanujan's last letter to Hardy in 1920 (see, e.g., [1]). Although mock theta function is not modular, it has a quasimodular property as was first shown by Watson [24]. An underlying intrinsic mathematical structure of these functions has recently been clarified by Zwegers in his thesis [27]; the mock theta function is a holomorphic part of the harmonic Maass form with weight  $1/2$ , which is a Jacobi form and an eigenfunction of the second-order differential operator. As an application, Bringmann and Ono resolved a rank-generating problem of integer partitions [4] by constructing the Poincaré–Maass series [3].

The purpose of this paper is to apply Zwegers' method to the  $\mathcal{N} = 4$  superconformal algebras at the general level- $k$  and construct a Jacobi form for each massless representation. We show that the construction of a Jacobi form is useful in decomposing elliptic genus in terms of irreducible superconformal representations. As an application, we will determine the elliptic genera of hyper-Kähler manifolds in complex four dimensions, in particular, the Hilbert scheme of points on  $K3$  surfaces  $K3^{[2]}$  and complex tori  $A^{[3]}$ .

This paper is organized as follows. In section 2, we briefly review the representation theory and character formulae of  $\mathcal{N} = 4$  SCA. We reconsider the level-1 case using Zwegers' formalism in section 3. In section 4, we study in detail the higher-level  $\mathcal{N} = 4$  SCFT and present elliptic genera of complex four-dimensional hyper-Kähler manifolds. The definitions of theta functions and numerical data of some expansion coefficients are relegated to appendices A and B.

## 2. Superconformal algebra and elliptic genus

### 2.1. Character of superconformal algebra

The  $\mathcal{N} = 4$  superconformal algebra with the central charge  $c = 6k$  contains a level- $k$  affine  $SU(2)$  algebra. The highest weight states are labeled by the conformal dimension  $h$  and isospin  $\ell$ , satisfying  $0 \leq \ell \leq k/2$  with  $\ell \in \mathbb{Z}/2$ . Due to the unitarity, we have a bound on conformal dimension in the R and NS sectors as

$$\begin{cases} \text{R :} & h \geq \frac{k}{4}, \\ \text{NS :} & h \geq \ell. \end{cases}$$

In each of these sectors, there exist two types of representations [8–10];

- massive (non-BPS) representation,

$$\begin{cases} \text{R : } & h > \frac{k}{4}, & \ell = \frac{1}{2}, 1, \dots, \frac{k}{2}, \\ \text{NS : } & h > \ell, & \ell = 0, \frac{1}{2}, \dots, \frac{k-1}{2}, \end{cases}$$

- massless (BPS) representation,

$$\begin{cases} \text{R : } & h = \frac{k}{4}, & \ell = 0, \frac{1}{2}, \dots, \frac{k}{2}, \\ \text{NS : } & h = \ell, & \ell = 0, \frac{1}{2}, \dots, \frac{k}{2}. \end{cases}$$

The character is a trace over the representation space

$$\text{ch}_{k,h,\ell}(z; \tau) = \text{Tr}_{\mathcal{H}} \left( e^{2\pi i z T_0^3} q^{L_0 - \frac{c}{24}} \right), \tag{2.1}$$

where  $q = e^{2\pi i \tau}$  with  $\tau \in \mathbb{H}$  as usual.  $L_0$  and  $T_0^3$  denote the zero mode of the Virasoro operator and the third component of the  $SU(2)$  current, respectively. The characters in the Ramond sector, for instance, are computed as [8, 9]

- massive character,

$$\text{ch}_{k,h,\ell}^R(z; \tau) = q^{h - \frac{\ell^2}{k+1} - \frac{k}{4}} \frac{[\theta_{10}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{k-1,\ell-\frac{1}{2}}(z; \tau), \tag{2.2}$$

- massless character,

$$\text{ch}_{k,\frac{k}{4},\ell}^R(z; \tau) = \frac{i}{\theta_{11}(2z; \tau)} \cdot \frac{[\theta_{10}(z; \tau)]^2}{[\eta(\tau)]^3} \sum_{\varepsilon=\pm 1} \sum_{m \in \mathbb{Z}} \varepsilon \frac{e^{4\pi i \varepsilon((k+1)m+\ell)z}}{(1 + e^{-2\pi i \varepsilon z} q^{-m})^2} q^{(k+1)m^2 + 2\ell m}. \tag{2.3}$$

See appendix A for definitions of theta functions.  $\chi_{k,\ell}(z; \tau)$  denotes the character of the spin  $\ell$  representation of  $SU(2)_k$  algebra. Note that the denominator in the massless character originates from the BPS condition that the supercharge annihilates the BPS states. If we ignore the denominator  $(1 + e^{-2\pi i \varepsilon z} q^{-m})^2$  in the massless character, it takes the same form as the massive character at  $h = k/4$ . We can check directly that the massless characters (2.3) satisfy a recursion relation,

$$\text{ch}_{k,\frac{k}{4},\ell}^R(z; \tau) + 2 \text{ch}_{k,\frac{k}{4},\ell-\frac{1}{2}}^R(z; \tau) + \text{ch}_{k,\frac{k}{4},\ell-1}^R(z; \tau) = q^{-\frac{\ell^2}{k+1}} \frac{[\theta_{10}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{k-1,\ell-\frac{1}{2}}(z; \tau). \tag{2.4}$$

Note that the right-hand side corresponds to the character of massive representation in the limit

$$\lim_{h \searrow \frac{k}{4}} \text{ch}_{k,h,\ell}^R(z; \tau) = q^{-\frac{\ell^2}{k+1}} \frac{[\theta_{10}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{k-1,\ell-\frac{1}{2}}(z; \tau),$$

and (2.4) shows how the non-BPS representation decomposes into a sum of BPS representations at the unitarity bound.

The characters in the NS sector are given from those in the R sector by the spectral flow as

$$\text{ch}_{k,h+\frac{\ell}{2}+\frac{k}{4},\frac{k}{2}-\ell}^{\text{NS}}(z; \tau) = q^{\frac{k}{4}} e^{2\pi i k z} \text{ch}_{k,h,\ell}^R\left(z + \frac{\tau}{2}; \tau\right). \tag{2.5}$$

Characters of other sectors are also defined by

$$\widetilde{\text{ch}}_{k,h,\ell}^R(z; \tau) = \text{ch}_{k,h,\ell}^R\left(z + \frac{1}{2}; \tau\right), \tag{2.6}$$

$$\widetilde{\text{ch}}_{k,h,\ell}^{\text{NS}}(z; \tau) = \text{ch}_{k,h,\ell}^{\text{NS}}\left(z + \frac{1}{2}; \tau\right). \tag{2.7}$$

2.2. Elliptic genus

When the superconformal field theory is interpreted as a nonlinear sigma model on the Calabi–Yau manifold  $X$  with a complex dimension  $c/3$ , the elliptic genus of  $X$  is identified with [25]

$$Z_X(z; \tau) = \text{Tr}_{\mathcal{H}^R \otimes \mathcal{H}^R} (-1)^F e^{2\pi i T_0^3 z} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \tag{2.8}$$

where  $(-1)^F = e^{\pi i (T_0^3 - \bar{T}_0^3)}$ , and  $\mathcal{H}^R$  denotes the Hilbert space of the Ramond sector. Due to the supersymmetry, the elliptic genus is independent of  $\bar{q}$ , and only the ground states contribute in the right-moving sector. The fundamental structure of the elliptic genus  $Z_X(z; \tau)$  is the modular property [11, 16]:

$$\begin{aligned} Z_X\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) &= e^{2\pi i \frac{c}{6} \frac{z^2}{\tau}} Z_X(z; \tau), \\ Z_X(z; \tau + 1) &= Z_X(z; \tau), \\ Z_X(z + 1; \tau) &= (-1)^{\frac{c}{3}} Z_X(z; \tau), \\ Z_X(z + \tau; \tau) &= (-1)^{\frac{c}{3}} q^{-\frac{c}{6}} e^{-2\pi i \frac{c}{3} z} Z_X(z; \tau). \end{aligned} \tag{2.9}$$

It is known that the elliptic genus at special values of  $z$  gives classical topological invariants of  $X$  [5]:

$$\begin{aligned} Z_X(z = 0; \tau) &= \chi_X, \\ Z_X\left(z = \frac{1}{2}; \tau\right) &= \sigma_X + \mathcal{O}(q), \\ q^{\frac{c}{12}} Z_X\left(z = \frac{1 + \tau}{2}; \tau\right) &= \widehat{A}_X + \mathcal{O}(q), \end{aligned} \tag{2.10}$$

where  $\chi_X$ ,  $\sigma_X$  and  $\widehat{A}_X$  are respectively the Euler characteristic, the Hirzebruch signature and the  $\widehat{A}$ -genus of  $X$ .

3. Level- $k = 1$  superconformal algebra revisited

The massless character in the  $\widetilde{R}$ -sector is read from (2.3) as

$$\text{ch}_{k=1, h=\frac{1}{4}, \ell=0}^{\widetilde{R}}(z; \tau) = \frac{i}{\theta_{11}(2z; \tau)} \cdot \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \sum_{m \in \mathbb{Z}} q^{2m^2} e^{8\pi i m z} \frac{1 + e^{2\pi i z} q^m}{1 - e^{2\pi i z} q^m}, \tag{3.1}$$

and the identity (2.4) reduces to

$$\text{ch}_{k=1, h=\frac{1}{4}, \ell=\frac{1}{2}}^{\widetilde{R}}(z; \tau) + 2 \text{ch}_{k=1, h=\frac{1}{4}, \ell=0}^{\widetilde{R}}(z; \tau) = q^{-\frac{1}{8}} \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3}. \tag{3.2}$$

The character (3.1) is known to be rewritten as [10]

$$\text{ch}_{k=1, h=\frac{1}{4}, \ell=0}^{\widetilde{R}}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \mu(z; \tau), \tag{3.3}$$

where

$$\mu(z; \tau) = \frac{i e^{\pi i z}}{\theta_{11}(z; \tau)} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n(n+1)} \frac{e^{2\pi i n z}}{1 - q^n e^{2\pi i z}}. \tag{3.4}$$

We reconsider the modular properties of  $\mu(z; \tau)$  using Zweegers' approach.

3.1. Harmonic Maass form

Zwegers defines the function [27]

$$\mu(u, v; \tau) = \frac{i e^{\pi i u}}{\theta_{11}(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-e^{2\pi i v})^n q^{\frac{1}{2}n(n+1)}}{1 - e^{2\pi i u} q^n}, \tag{3.5}$$

which is known as the Lerch sum (or generalized Lambert series).  $\mu(z; \tau)$  of (3.4) and  $\mu(u, v; \tau)$  are simply related as  $\mu(u = z, v = z; \tau) = \mu(z; \tau)$ . Zwegers showed that, under the  $S$ -transformation,  $\mu(u, v; \tau)$  behaves as

$$\mu(u, v; \tau) + \sqrt{\frac{i}{\tau}} e^{\pi i \frac{(u-v)^2}{\tau}} \mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = \frac{1}{2} M(u - v; \tau), \tag{3.6}$$

where  $M(x; \tau)$  is the Mordell integral [18] defined by

$$M(x; \tau) = \int_{-\infty}^{\infty} \frac{e^{\pi i \tau z^2 - 2\pi x z}}{\cosh(\pi z)} dz. \tag{3.7}$$

He further introduced a non-holomorphic partner of  $\mu(u, v; \tau)$  by

$$R(z; \tau) = \sum_{n \in \mathbb{Z}} (-1)^n \left[ \operatorname{sgn}\left(n + \frac{1}{2}\right) - E\left(\left(n + \frac{1}{2} + \frac{\Im z}{\Im \tau}\right) \sqrt{2\Im \tau}\right) \right] e^{-2\pi i(n+\frac{1}{2})z} q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \tag{3.8}$$

where  $E(z)$  is the error function defined by

$$E(z) = 2 \int_0^z e^{-\pi u^2} du = 1 - \operatorname{erfc}(\sqrt{\pi}z). \tag{3.9}$$

The non-holomorphic function is shown to fulfil the  $S$ -transformation

$$R(u; \tau) + \sqrt{\frac{i}{\tau}} e^{\pi i \frac{u^2}{\tau}} R\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) = M(u; \tau). \tag{3.10}$$

Zwegers combined these two functions to define

$$\widehat{\mu}(u, v; \tau) = \mu(u, v; \tau) - \frac{1}{2} R(u - v; \tau), \tag{3.11}$$

as a completion of the Lerch sum  $\mu(u, v; \tau)$ . The Mordell integral  $M(u; \tau)$  disappears in the  $S$ -transformation of  $\widehat{\mu}(u, v; \tau)$ , and it behaves like a two-variable Jacobi form with weight  $1/2$  [11]:

$$\begin{aligned} \widehat{\mu}(u, v; \tau) &= -\sqrt{\frac{i}{\tau}} e^{\pi i \frac{(u-v)^2}{\tau}} \widehat{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) \\ &= e^{\pi i/4} \widehat{\mu}(u, v; \tau + 1). \end{aligned} \tag{3.12}$$

In view of (3.4) and (3.5), we see that the massless character (3.3) is proportional to the holomorphic part of the real analytic modular form

$$\widehat{\mu}(z; \tau) = \mu(z; \tau) - \frac{1}{2} R(0; \tau), \tag{3.13}$$

whose transformation formulae are

$$\begin{aligned} \widehat{\mu}(z; \tau) &= -\sqrt{\frac{i}{\tau}} \widehat{\mu}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \\ \widehat{\mu}(z; \tau + 1) &= e^{-\frac{1}{4}\pi i} \widehat{\mu}(z; \tau), \\ \widehat{\mu}(z + 1; \tau) &= \widehat{\mu}(z + \tau; \tau) = \widehat{\mu}(z; \tau). \end{aligned} \tag{3.14}$$

It is possible to show that the non-holomorphic part  $R(0; \tau)$  in (3.13) is an incomplete period integral of the modular form  $[\eta(\tau)]^3$  with weight  $3/2$ :

$$iR(0; \tau) = \int_{-\bar{\tau}}^{i\infty} \frac{[\eta(z)]^3}{\sqrt{\frac{z+\tau}{i}}} dz, \tag{3.15}$$

which shows that the massless character (3.3) of the level-1 superconformal algebra has a modular form  $[\eta(\tau)]^3$  as a ‘shadow’ in the sense of Zagier [26]. We then see that the real analytic function  $\widehat{\mu}(z; \tau)$  satisfies

$$\frac{\partial}{\partial \bar{\tau}} \widehat{\mu}(z; \tau) = \frac{i}{2} \frac{[\eta(-\bar{\tau})]^3}{\sqrt{2\Im\tau}}. \tag{3.16}$$

To conclude, we see that the function  $\widehat{\mu}(z; \tau)$  is a harmonic Maass form being a solution of the differential equation

$$(\Im\tau)^{\frac{3}{2}} \frac{\partial}{\partial \tau} \sqrt{\Im\tau} \frac{\partial}{\partial \bar{\tau}} \widehat{\mu}(z; \tau) = 0, \tag{3.17}$$

which, with  $\tau = u + iv$ , reduces to

$$\left[ -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \frac{i}{2} v \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right] \widehat{\mu}(z; \tau) = 0. \tag{3.18}$$

We note that the formula under the  $S$ -transformation (3.6) gives [10]

$$\frac{\mu}{\eta}(z; \tau) + \frac{\mu}{\eta}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \frac{1}{\eta(\tau)} \int_{-\infty}^{\infty} \frac{e^{\pi i \tau x^2}}{2 \cosh(\pi x)} dx. \tag{3.19}$$

### 3.2. Character decomposition of elliptic genus

Let the function  $J(z; w; \tau)$  be

$$\begin{aligned} J(z; w; \tau) &= \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} (\widehat{\mu}(z; \tau) - \widehat{\mu}(w; \tau)) \\ &= \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} (\mu(z; \tau) - \mu(w; \tau)) \\ &= \text{ch}_{k=1, h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) - \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \mu(w; \tau), \end{aligned} \tag{3.20}$$

where the second equality follows from the  $z$ -independence of  $R(0; \tau)$  (3.13). We see that

$$J(w; w; \tau) = 0, \tag{3.21}$$

and that  $J(0; w; \tau) = 1$ . By use of (3.14), we obtain the transformation formulae as follows:

$$\begin{aligned} J(z; w; \tau) &= e^{-2\pi i \frac{z^2}{\tau}} J\left(\frac{z}{\tau}; \frac{w}{\tau}; -\frac{1}{\tau}\right), \\ J(z+1; w; \tau) &= J(z; w; \tau+1) = J(z; w+\tau; \tau) = J(z; w; \tau), \\ J(z+\tau; w; \tau) &= q^{-1} e^{-4\pi i z} J(z; w; \tau). \end{aligned} \tag{3.22}$$

The second equality of (3.20) indicates that the function  $J(z; w; \tau)$  is a holomorphic function; when we fix  $w$  to be the specific values  $w = \frac{1}{2}$ ,  $\frac{1+\tau}{2}$ , and  $\frac{\tau}{2}$ , those modular properties show

$$\begin{aligned} J\left(z; \frac{1}{2}; \tau\right) &= \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^2, & J\left(z; \frac{1+\tau}{2}; \tau\right) &= \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^2, \\ J\left(z; \frac{\tau}{2}; \tau\right) &= \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^2. \end{aligned} \tag{3.23}$$

Lerch sums introduced in [10] are given by the values of  $\mu(w; \tau)$  at  $w = 1/2, 1 + \tau/2, \tau/2$ :

$$\begin{aligned}
 h_2(\tau) &\equiv \frac{\mu\left(\frac{1}{2}; \tau\right)}{\eta} = \frac{1}{\eta(\tau)\theta_{10}(0; \tau)} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}n(n+1)}}{1 + q^n}, \\
 h_3(\tau) &\equiv \frac{\mu\left(\frac{1+\tau}{2}; \tau\right)}{\eta} = \frac{1}{\eta(\tau)\theta_{00}(0; \tau)} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{1}{2}n^2 - \frac{1}{8}}}{1 + q^{n - \frac{1}{2}}}, \\
 h_4(\tau) &\equiv \frac{\mu\left(\frac{\tau}{2}; \tau\right)}{\eta} = \frac{1}{\eta(\tau)\theta_{01}(0; \tau)} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n^2 - \frac{1}{8}}}{1 + q^{n - \frac{1}{2}}}.
 \end{aligned}
 \tag{3.24}$$

We then obtain the decomposition formula for the massless character

$$\begin{aligned}
 \text{ch}_{k=1, h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) &= \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^2 + h_2(\tau) \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^2}, \\
 &= \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^2 + h_3(\tau) \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^2}, \\
 &= \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^2 + h_4(\tau) \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^2}.
 \end{aligned}
 \tag{3.25}$$

Identities (3.25) are used to express the elliptic genus of the K3 surface in terms of irreducible representations of  $\mathcal{N} = 4$  algebra. The elliptic genus of the K3 surface is known to be given by [5, 16]

$$Z_{K3}(z; \tau) = 8 \left[ \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^2 + \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^2 + \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^2 \right].
 \tag{3.26}$$

Using (3.25), we can rewrite the elliptic genus as

$$Z_{K3}(z; \tau) = 24 \text{ch}_{k=1, h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) - 8 \left[ \frac{[\theta_{11}(z; \tau)]^2}{\eta(\tau)} \right] \sum_{a=2,3,4} h_a(\tau).$$

We note that

$$\begin{aligned}
 8 \sum_{w \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}} \mu(w; \tau) &= 8\eta(\tau) \sum_{a=2,3,4} h_a(\tau) \\
 &= q^{-\frac{1}{8}} \left( 2 - \sum_{n \geq 1} A_n q^n \right),
 \end{aligned}$$

where  $A_n$  are positive integers:

$n$	1	2	3	4	5	6	7	8	$\dots$
$A_n$	90	462	1540	4554	11592	27830	61686	131100	$\dots$

(3.27)

Using relation (3.2), we obtain

$$Z_{K3}(z; \tau) = 20 \text{ch}_{1, \frac{1}{4}, 0}^{\tilde{R}}(z; \tau) - 2 \text{ch}_{1, \frac{1}{4}, \frac{1}{2}}^{\tilde{R}}(z; \tau) + \sum_{n \geq 1} A_n \text{ch}_{1, n + \frac{1}{4}, \frac{1}{2}}^{\tilde{R}}(z; \tau),
 \tag{3.28}$$

where the first two terms are massless characters and the last term gives an infinite sum of massive characters. Note that the right-moving sector in the elliptic genus (2.8) gives a degeneracy factor  $2\ell + 1$  for isospin  $\ell$  representation. Hence, the net multiplicity of isospin 1/2 massless representation in (3.28) equals 1. Since isospin 1/2 representation in the R sector flows to isospin 0 in the NS sector, (3.28) has the multiplicity 1 for isospin 0 or vacuum representation in the NS sector.



#### 4. Higher-level superconformal algebra

The massless character at the level- $k$  with isospin 0 in the  $\tilde{R}$ -sector is read from (2.3) as

$$\text{ch}_{k, \frac{k}{4}, \ell=0}^{\tilde{R}}(z; \tau) = \frac{i}{\theta_{11}(2z; \tau)} \cdot \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \sum_{m \in \mathbb{Z}} e^{4\pi i(k+1)mz} q^{(k+1)m^2} \frac{1 + e^{2\pi iz} q^m}{1 - e^{2\pi iz} q^m}, \quad (4.1)$$

and the recursion relation (2.4) of massless characters becomes

$$\begin{aligned} \text{ch}_{k, \frac{k}{4}, \ell}^{\tilde{R}}(z; \tau) + 2 \text{ch}_{k, \frac{k}{4}, \ell-\frac{1}{2}}^{\tilde{R}}(z; \tau) + \text{ch}_{k, \frac{k}{4}, \ell-1}^{\tilde{R}}(z; \tau) \\ = (-1)^{2\ell+1} q^{-\frac{\ell^2}{k+1}} \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{k-1, \ell-\frac{1}{2}}(z; \tau). \end{aligned} \quad (4.2)$$

The modular transformation properties of (4.1) are studied [7] by use of the formula of Miki [17] and also the higher-level Appell function [20, 23]. We shall show that these massless characters are the mock theta functions whose ‘shadow’ [26] is  $\Psi_{k+1}^{(a)}(\tau)$  (A.10) related to the affine  $SU(2)$  characters at the level- $k$ . It is also related to the colored Jones polynomial for torus link  $T(2, 2P)$  [12], and is a building block for the  $SU(2)$  Witten–Reshetikhin–Turaev invariant for the Seifert manifolds associated with the ADE singularities [13, 14].

##### 4.1. Setup of Zwegers

We first review the results of Zwegers [27]. We introduce the Lerch sum as

$$f_P(u, z; \tau) = \sum_{n \in \mathbb{Z}} \frac{q^{Pn^2} e^{4\pi i Pnz}}{1 - q^n e^{2\pi i(z-u)}}, \quad (4.3)$$

where  $P \in \mathbb{Z}_{>0}$ , and it is related to the level- $k$  of the superconformal algebra as  $k = P - 1$ . Zwegers proved the following formulae:

$$\begin{aligned} f_P(u+1, z; \tau) &= f_P(u, z; \tau), \\ f_P(u, z; \tau) - q^{-P} e^{-4\pi i Pu} f_P(u+\tau, z; \tau) &= \sum_{a=0}^{2P-1} q^{-\frac{a^2}{4P}} e^{-2\pi i au} \vartheta_{P,a}(z; \tau), \\ f_P(u, z; \tau+1) &= f_P(u, z; \tau), \end{aligned} \quad (4.4)$$

$$f_P(u, z; \tau) - \frac{1}{\tau} e^{2\pi i P \frac{u^2-z^2}{\tau}} f_P\left(\frac{u}{\tau}, \frac{z}{\tau}; \frac{-1}{\tau}\right) = \sum_{a=0}^{2P-1} M_{P,a}(u; \tau) \vartheta_{P,a}(z; \tau),$$

where  $\vartheta_{P,a}(z; \tau)$  are level- $P$  theta functions (A.4), and  $M_{P,a}(u; \tau)$  has a form of the Mordell integral

$$M_{P,a}(u; \tau) = i e^{-\pi i \frac{a^2}{2P} \tau - 2\pi i au} \int_{\mathbb{R}-i0} \frac{e^{2\pi i P \tau x^2 - 2\pi(2Pu+a\tau)x}}{1 - e^{2\pi x}} dx. \quad (4.5)$$

As a completion of  $f_P(u, z; \tau)$ , we define  $\widehat{f}_P(u, z; \tau)$  by

$$\widehat{f}_P(u, z; \tau) = f_P(u, z; \tau) - \frac{1}{2} \sum_{a \bmod 2P} R_{P,a}(u; \tau) \vartheta_{P,a}(z; \tau), \quad (4.6)$$

where the non-holomorphic partner  $R_{P,a}(u; \tau)$  is given by

$$R_{P,a}(u; \tau) = \sum_{\substack{n \in \mathbb{Z} \\ n \equiv a \bmod 2P}} \left[ \text{sgn}\left(n + \frac{1}{2}\right) - E\left(\left(n + 2P \frac{\Im u}{\Im \tau}\right) \sqrt{\frac{\Im \tau}{P}}\right) \right] q^{-\frac{n^2}{4P}} e^{-2\pi i nu}. \quad (4.7)$$

We then find that the function  $\widehat{f}_P(u, z; \tau)$  has a modular property similar to a two-variable Jacobi form with weight 1:

$$\begin{aligned} \widehat{f}_P(u, z; \tau + 1) &= \widehat{f}_P(u, z; \tau), \\ \widehat{f}_P\left(\frac{u}{\tau}, \frac{z}{\tau}; -\frac{1}{\tau}\right) &= \tau e^{2\pi i P \frac{z^2 - u^2}{\tau}} \widehat{f}_P(u, z; \tau). \end{aligned} \tag{4.8}$$

4.2. Generalized Mordell integral

We study the period function of the weight-3/2 vector-valued modular form  $\{\Psi_P^{(a)}(\tau) | 0 < a < P\}$  defined in (A.9):

$$M_P^{(a)}(\tau) = \int_0^{i\infty} \frac{\Psi_P^{(a)}(z)}{\sqrt{\frac{z+\tau}{i}}} dz. \tag{4.9}$$

We substitute (A.10) for the above integrand, and apply an identity [27, lemma 1.18]

$$\int_{-\infty}^{\infty} \frac{e^{\pi i r w^2}}{w + ir} dw = -\pi r \int_0^{i\infty} \frac{e^{\pi i r^2 z}}{\sqrt{\frac{z+\tau}{i}}} dz. \tag{4.10}$$

Then the period function is rewritten as

$$M_P^{(a)}(\tau) = -\frac{\sqrt{2P}}{2\pi} \int_{\mathbb{R}} dz e^{\pi i r \frac{z^2}{2P}} \sum_{k \in \mathbb{Z}} \left( \frac{1}{z + i(2Pk + a)} - \frac{1}{z + i(2Pk - a)} \right).$$

When we make use of an infinite series expansion

$$\tanh\left(\frac{\pi}{2}x\right) = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2},$$

we find that the period function is written in the form of the generalized Mordell integral as

$$\begin{aligned} M_P^{(a)}(\tau) &= -\frac{1}{2\sqrt{2P}} \int_{\mathbb{R}} dz e^{\pi i r \frac{z^2}{2P}} \left[ \tanh\left(\frac{\pi(z + i(a - P))}{2P}\right) - \tanh\left(\frac{\pi(z - i(a - P))}{2P}\right) \right] \\ &= \frac{i}{\sqrt{2P}} \int_{\mathbb{R}} dz e^{\pi i r \frac{z^2}{2P}} \frac{\sin\left(\frac{P-a}{P}\pi\right)}{\cosh\left(\frac{z}{P}\pi\right) + \cos\left(\frac{P-a}{P}\pi\right)}. \end{aligned} \tag{4.11}$$

This form of the Mordell integral has also been noted in [23]. These integrals have another form,

$$M_P^{(a)}(\tau) = i\sqrt{\frac{i}{\tau}} \int_{-\infty}^{\infty} dx e^{-\pi i \frac{x^2}{2P\tau}} \frac{\sinh\left(\pi \frac{P-a}{P}x\right)}{\sinh(\pi x)}, \tag{4.12}$$

which is known to generate the Ohtsuki-type invariant for the torus link [12]. Note that the generalized Mordell integral (4.11) is also used in the studies of the integer partition [2]. Expression (4.9) of the Mordell integral as a period integral easily proves the S-transformation formula

$$\sum_{b=1}^{P-1} \mathbf{S}(P)_{ab} M_P^{(b)}(\tau) = \sqrt{\frac{i}{\tau}} M_P^{(a)}\left(-\frac{1}{\tau}\right), \tag{4.13}$$

when we substitute (A.11) for (4.9).

Relationship of  $M_P^{(a)}(\tau)$  with Zwegers' function (4.5)  $M_{P,a}(u; \tau)$  is given as follows: from (4.11), we compute as

$$\begin{aligned} M_P^{(a)}(\tau) &= \frac{e^{-\pi i \tau \frac{(a-P)^2}{2P}}}{2\sqrt{2P}} \int_{\mathbb{R}} dw \tanh\left(\frac{\pi w}{2P}\right) e^{\pi i \tau \frac{w^2}{2P}} \left( e^{\pi \tau \frac{P-a}{P} w} - e^{-\pi \tau \frac{P-a}{P} w} \right) \\ &= \frac{e^{-\pi i \frac{a^2}{2P} \tau}}{\sqrt{2P}} \left[ \int_{\mathbb{R}+i0} \frac{e^{\pi i \tau \frac{z^2}{2P} - \pi a \tau \frac{z}{P}}}{1 - e^{-\pi \frac{z}{P}}} dz - \int_{\mathbb{R}-i0} \frac{e^{\pi i \tau \frac{z^2}{2P} - \pi a \tau \frac{z}{P}}}{1 - e^{\pi \frac{z}{P}}} dz \right]. \end{aligned}$$

Thus we obtain

$$M_P^{(a)}(\tau) = i\sqrt{2P}[M_{P,a}(0; \tau) - M_{P,-a}(0; \tau)]. \tag{4.14}$$

### 4.3. Non-holomorphic function

We define a non-holomorphic function,

$$R_P^{(a)}(\tau) = \int_{-\bar{\tau}}^{i\infty} \frac{\Psi_P^{(a)}(z)}{\sqrt{\frac{z+\tau}{i}}} dz, \tag{4.15}$$

as a generalization of (3.15). This integral expression shows that

$$\frac{\partial}{\partial \bar{\tau}} R_P^{(a)}(\tau) = \frac{\Psi_P^{(a)}(-\bar{\tau})}{\sqrt{2\Im \tau}}. \tag{4.16}$$

By integrating each summand of  $\Psi_P^{(a)}(z)$  after the substitution of (A.10) into (4.15), we find

$$R_P^{(a)}(\tau) = i\sqrt{2P}R_{P,a}(0; \tau), \tag{4.17}$$

where  $R_{P,a}(z; \tau)$  is Zwegers'  $R$ -function defined in (4.7).

Substituting the modular transformation law (A.11) for  $\Psi_P^{(a)}(z)$  in (4.15), we find the transformation law for  $R_P^{(a)}(\tau)$  as follows:

$$\begin{aligned} \sqrt{\frac{i}{\tau}} \sum_{b=1}^{P-1} \mathbf{S}(P)_{ab} R_P^{(b)}\left(-\frac{1}{\tau}\right) + R_P^{(a)}(\tau) &= M_P^{(a)}(\tau), \\ R_P^{(a)}(\tau + 1) &= e^{-\pi i \frac{a^2}{2P}} R_P^{(a)}(\tau), \end{aligned} \tag{4.18}$$

where  $M_P^{(a)}(\tau)$  is the Mordell integral defined in (4.9). Using the level- $P$   $\vartheta$ -function (A.4), the  $S$ -transformation formula is rewritten as

$$\begin{aligned} \sum_{a=1}^{P-1} M_P^{(a)}(\tau) (\vartheta_{P,a} - \vartheta_{P,-a})(z; \tau) &= \left[ \sum_{a=1}^{P-1} R_P^{(a)}(\tau) (\vartheta_{P,a} - \vartheta_{P,-a})(z; \tau) \right] \\ &\quad - \frac{1}{\tau} e^{-2\pi i \frac{P}{\tau} z^2} \left[ \sum_{a=1}^{P-1} R_P^{(a)}\left(-\frac{1}{\tau}\right) (\vartheta_{P,a} - \vartheta_{P,-a})\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) \right]. \end{aligned} \tag{4.19}$$

### 4.4. Modular transformation of massless characters

The massless character (4.1) is identified with an anti-symmetric part of the Lerch sum (4.3). Using (4.4) and (4.14), we find

$$\begin{aligned} \frac{1}{i\sqrt{2P}} \sum_{a=1}^{P-1} M_P^{(a)}(\tau) [\vartheta_{P,a}(z; \tau) - \vartheta_{P,-a}(z; \tau)] &= \left[ \sum_{n \in \mathbb{Z}} q^{Pn^2} e^{4\pi i Pnz} \frac{1 + q^n e^{2\pi iz}}{1 - q^n e^{2\pi iz}} \right] \\ &\quad - \frac{1}{\tau} e^{-2\pi i P \frac{z^2}{\tau}} \left[ \sum_{n \in \mathbb{Z}} \tilde{q}^{Pn^2} e^{4\pi i Pn \frac{z}{\tau}} \frac{1 + \tilde{q}^n e^{2\pi i \frac{z}{\tau}}}{1 - \tilde{q}^n e^{2\pi i \frac{z}{\tau}}} \right], \end{aligned}$$

where  $\tilde{q} = e^{-2\pi i/\tau}$ . We thus obtain a formula which is a generalization of (3.19):

$$\begin{aligned}
 F_P(z; \tau) + e^{-2\pi i(P-2)\frac{z}{\tau}} F_P\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) &= \frac{1}{i\sqrt{2P}} \sum_{a=1}^{P-1} \frac{M_P^{(a)}(\tau)}{\eta(\tau)} \cdot \frac{\vartheta_{P,a} - \vartheta_{P,-a}}{\vartheta_{2,1} - \vartheta_{2,-1}}(z; \tau) \\
 &= \frac{1}{i\sqrt{2P}} \sum_{a=1}^{P-1} \frac{M_P^{(a)}(\tau)}{\eta(\tau)} \chi_{P-2, \frac{a-1}{2}}(z; \tau),
 \end{aligned}
 \tag{4.20}$$

where we have defined

$$F_P(z; \tau) = \frac{i}{\eta(\tau)\theta_{11}(2z; \tau)} \sum_{n \in \mathbb{Z}} q^{Pn^2} e^{4\pi i Pnz} \frac{1 + q^n e^{2\pi iz}}{1 - q^n e^{2\pi iz}}.
 \tag{4.21}$$

We see that (4.20) with  $P = 2$  coincides with (3.19), and the massless character (4.1) is related to  $F_P(z; \tau)$  as

$$\text{ch}_{k, \frac{k}{4}, \ell=0}^{\tilde{R}}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} F_{k+1}(z; \tau).
 \tag{4.22}$$

#### 4.5. Harmonic Maass form

Looking at the  $S$ -transformation (4.20) of the massless character, we see that the Mordell integrals can be compensated with those from (4.19). By setting  $C_P(z; \tau)$  to be the massless character (4.1) for notational simplicity,

$$\begin{aligned}
 C_P(z; \tau) &= \text{ch}_{k=P-1, h=\frac{P-1}{4}, \ell=0}^{\tilde{R}}(z; \tau) \\
 &= \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \frac{i}{\theta_{11}(2z; \tau)} \sum_{n \in \mathbb{Z}} q^{Pn^2} e^{4\pi i Pnz} \frac{1 + q^n e^{2\pi iz}}{1 - q^n e^{2\pi iz}},
 \end{aligned}
 \tag{4.23}$$

we introduce a (real analytic) Jacobi as

$$\widehat{C}_P(z; \tau) = C_P(z; \tau) - \frac{1}{i\sqrt{2P}} \sum_{a=1}^{P-1} R_P^{(a)}(\tau) B_P^{(a)}(z; \tau).
 \tag{4.24}$$

Here the basis function  $B_P^{(a)}(z; \tau)$  is defined as

$$B_P^{(a)}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \cdot \frac{\vartheta_{P,a} - \vartheta_{P,-a}}{\vartheta_{2,1} - \vartheta_{2,-1}}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{P-2, \frac{a-1}{2}}(z; \tau).
 \tag{4.25}$$

The modular transformation formulae of  $\widehat{C}_P(z; \tau)$  are summarized as follows:

$$\begin{aligned}
 \widehat{C}_P(z; \tau) &= e^{-2\pi i(P-1)\frac{z}{\tau}} \widehat{C}_P\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \\
 \widehat{C}_P(z; \tau + 1) &= \widehat{C}_P(z + 1; \tau) = \widehat{C}_P(z; \tau), \\
 \widehat{C}_P(z + \tau; \tau) &= q^{-(P-1)} e^{-4\pi i(P-1)z} \widehat{C}_P(z; \tau).
 \end{aligned}
 \tag{4.26}$$

We note that the Fourier expansions have the form

$$\begin{aligned}
 C_P(z; \tau) &= 1 + (e^{\pi iz} - e^{-\pi iz})^2 (e^{2\pi iz} + e^{-2\pi iz})q + \dots, \\
 B_P^{(a)}(z; \tau) &= -(e^{\pi iz} - e^{-\pi iz})^2 I_{\frac{a-1}{2}}(z) q^{\frac{a^2}{4P}} + \dots,
 \end{aligned}
 \tag{4.27}$$

where  $I_{\frac{a}{2}}(z)$  is associated with the Chebyshev polynomial of the second kind:

$$I_{\frac{a}{2}}(z) = \sum_{n=-\frac{a}{2}}^{\frac{a}{2}} e^{4\pi inz}.$$

To construct an eigenfunction of the differential operator (3.17) at a higher level, we introduce a holomorphic function

$$H_P^{(a)}(z; \tau) = \frac{\text{Wr}[B_P^{(a)} \rightarrow C_P]}{\text{Wr}[B_P^{(1)}, \dots, B_P^{(P-1)}]}(z; \tau), \quad (4.28)$$

whose completion is given by

$$\widehat{H}_P^{(a)}(z; \tau) = \frac{\text{Wr}[B_P^{(a)} \rightarrow \widehat{C}_P]}{\text{Wr}[B_P^{(1)}, \dots, B_P^{(P-1)}]}(z; \tau). \quad (4.29)$$

Here  $\text{Wr}[\phi_1, \dots, \phi_n](z; \tau)$  denotes the Wronskian with respect to  $z$ :

$$\text{Wr}[\phi_1, \dots, \phi_n](z; \tau) = \begin{vmatrix} \phi_1(z; \tau) & \dots & \phi_n(z; \tau) \\ \vdots & \ddots & \vdots \\ \frac{d^{n-1}\phi_1}{dz^{n-1}}(z; \tau) & \dots & \frac{d^{n-1}\phi_n}{dz^{n-1}}(z; \tau) \end{vmatrix}$$

and  $\text{Wr}[B_P^{(a)} \rightarrow \widehat{C}_P](z; \tau)$  means the Wronskian  $\text{Wr}[B_P^{(1)}, \dots, B_P^{(P-1)}](z; \tau)$  with  $B_P^{(a)}(z; \tau)$  replaced by  $\widehat{C}_P(z; \tau)$ . From (4.24) and (4.16), we have

$$\begin{aligned} \frac{\partial}{\partial \bar{\tau}} \widehat{H}_P^{(a)}(z; \tau) &= \frac{i}{\sqrt{2P}} \cdot \frac{\partial R_P^{(a)}(\tau)}{\partial \bar{\tau}} \\ &= \frac{i}{\sqrt{2P}} \cdot \frac{\Psi_P^{(a)}(-\bar{\tau})}{\sqrt{2\Im\tau}}. \end{aligned} \quad (4.30)$$

As a result, we have

$$(\Im\tau)^{\frac{3}{2}} \frac{\partial}{\partial \tau} \sqrt{\Im\tau} \frac{\partial}{\partial \bar{\tau}} \widehat{H}_P^{(a)}(z; \tau) = 0,$$

which, with  $\tau = u + iv$ , reduces to

$$\left[ -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \frac{iv}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right] \widehat{H}_P^{(a)}(z; \tau) = 0. \quad (4.31)$$

We can check the modular transformation formulae as

$$\begin{aligned} \widehat{H}_P^{(a)}(z; \tau) &= -\sqrt{\frac{i}{\tau}} \sum_{b=1}^{P-1} \mathbf{S}(P)_{ab} \widehat{H}_P^{(b)}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \\ \widehat{H}_P^{(a)}(z; \tau + 1) &= e^{-\frac{\sigma^2}{2P}\pi i} \widehat{H}_P^{(a)}(z; \tau), \\ \widehat{H}_P^{(a)}(z + 1; \tau) &= \widehat{H}_P^{(a)}(z + \tau; \tau) = \widehat{H}_P^{(a)}(z; \tau). \end{aligned} \quad (4.32)$$

Thus recalling [21], we find that these are the harmonic Maass form with weight 1/2 on  $\Gamma(4P)$ .

#### 4.6. Character decomposition

We define an element of the  $(P - 1) \times (P - 1)$  matrix  $\mathbf{B}_P(\mathbf{w}; \tau)$  by

$$(\mathbf{B}_P(\mathbf{w}; \tau))_{ab} = B_P^{(b)}(w_a; \tau), \quad (4.33)$$

where  $\{w_1, w_2, \dots, w_{P-1}\}$  are coordinates on the torus. As a higher-level generalization of (3.20), we define  $J_P(z; w_1, \dots, w_{P-1}; \tau)$  by

$$J_P(z; w_1, \dots, w_{P-1}; \tau) = \widehat{C}_P(z; \tau) - \sum_{a=1}^{P-1} \sum_{b=1}^{P-1} (\mathbf{B}_P(\mathbf{w}; \tau)^{-1})_{ab} \widehat{C}_P(w_b; \tau) B_P^{(a)}(z; \tau) \quad (4.34)$$

$$= C_P(z; \tau) - \sum_{a=1}^{P-1} \sum_{b=1}^{P-1} (\mathbf{B}_P(\mathbf{w}; \tau)^{-1})_{ab} C_P(w_b; \tau) B_P^{(a)}(z; \tau), \quad (4.35)$$

where the second equality follows from the  $z$ -independence of the function  $R_P^{(a)}(\tau)$  (4.24). The function  $J_P(z; w_1, \dots, w_{P-1}; \tau)$  has a holomorphic  $q$ -series as seen from the second equality, and it has a good behavior under modular transformations. Explicitly, we have

$$\begin{aligned} J_P(z; w_1, \dots, w_{P-1}; \tau) &= e^{-2\pi i(P-1)\frac{z}{\tau}} J_P\left(\frac{z}{\tau}; \frac{w_1}{\tau}, \dots, \frac{w_{P-1}}{\tau}; -\frac{1}{\tau}\right), \\ J_P(z+1; w_1, \dots, w_{P-1}; \tau) &= J_P(z; w_1, \dots, w_a+1, \dots, w_{P-1}; \tau) \\ &= J_P(z; w_1, \dots, w_a+\tau, \dots, w_{P-1}; \tau) \\ &= J_P(z; w_1, \dots, w_{P-1}; \tau+1) \\ &= J_P(z; w_1, \dots, w_{P-1}; \tau), \end{aligned} \quad (4.36)$$

$$J_P(z+\tau; w_1, \dots, w_{P-1}; \tau) = q^{-(P-1)} e^{-4\pi i(P-1)z} J_P(z; w_1, \dots, w_{P-1}; \tau).$$

By construction, the function  $J_P(z; w_1, \dots, w_{P-1}; \tau)$  vanishes at  $z = w_a$  for  $a = 1, \dots, P-1$ ,

$$J_P(w_a; w_1, \dots, w_{P-1}; \tau) = 0, \quad (4.37)$$

and we also have

$$J_P(0; w_1, \dots, w_{P-1}; \tau) = 1. \quad (4.38)$$

4.6.1. *Degenerate case.* We consider the completely degenerate configuration,  $w_1 = w_2 = \dots = w_{P-1}$ ; we set

$$J_P(z; w; \tau) = \lim_{\substack{w_a \rightarrow w \\ \forall a}} J_P(z; w_1, \dots, w_{P-1}; \tau). \quad (4.39)$$

We then obtain from (4.34)

$$J_P(z; w; \tau) = \widehat{C}_P(z; \tau) - \sum_{a=1}^{P-1} \widehat{H}_P^{(a)}(w; \tau) B_P^{(a)}(z; \tau) \quad (4.40)$$

$$= C_P(z; \tau) - \sum_{a=1}^{P-1} H_P^{(a)}(w; \tau) B_P^{(a)}(z; \tau), \quad (4.41)$$

where  $\widehat{H}_P^{(a)}(z; \tau)$  is the harmonic Maass form of (4.29), and  $H_P^{(a)}(z; \tau)$  is its holomorphic part (4.28). The second line above gives a decomposition of a massless character  $C_P(z; \tau)$  into a Jacobi form  $J_P(z; w; \tau)$  and a sum over basis functions  $B_P^{(a)}(z; \tau)$ . Note that  $J_P(z; w; \tau)$  has  $2(P-1)$  th order zero at  $z = w$ . Its transformation formulae are given as

$$\begin{aligned} J_P(z; w; \tau) &= e^{-2\pi i(P-1)\frac{z}{\tau}} J_P\left(\frac{z}{\tau}; \frac{w}{\tau}; -\frac{1}{\tau}\right), \\ J_P(z+1; w; \tau) &= J_P(z; w+1; \tau) \\ &= J_P(z; w+\tau; \tau) \\ &= J_P(z; w; \tau+1) = J_P(z; w; \tau), \\ J_P(z+\tau; w; \tau) &= q^{-(P-1)} e^{-4\pi i(P-1)z} J_P(z; w; \tau). \end{aligned} \quad (4.42)$$

When we set  $w$  to special values  $\{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}$ , these formulae give the following identifications:

$$\begin{aligned} J_P\left(z; \frac{1}{2}; \tau\right) &= \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^{2(P-1)}, & J_P\left(z; \frac{1+\tau}{2}; \tau\right) &= \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^{2(P-1)}, \\ J_P\left(z; \frac{\tau}{2}; \tau\right) &= \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^{2(P-1)}. \end{aligned} \quad (4.43)$$

Hence we have

$$\text{ch}_{k=P-1, h=\frac{P-1}{4}, \ell=0}^{\tilde{R}}(z; \tau) = \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^{2(P-1)} + \sum_{a=1}^{P-1} H_P^{(a)}\left(\frac{1}{2}; \tau\right) B_P^{(a)}(z; \tau) \tag{4.44}$$

$$= \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^{2(P-1)} + \sum_{a=1}^{P-1} H_P^{(a)}\left(\frac{1+\tau}{2}; \tau\right) B_P^{(a)}(z; \tau) \tag{4.45}$$

$$= \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^{2(P-1)} + \sum_{a=1}^{P-1} H_P^{(a)}\left(\frac{\tau}{2}; \tau\right) B_P^{(a)}(z; \tau). \tag{4.46}$$

4.7. Elliptic genera

Character decomposition above can be used to rewrite the elliptic genera of hyper-Kähler manifolds  $X_k$  with  $\dim_{\mathbb{C}} X_k = 2k, k \geq 1$ .

We note that the elliptic genera for hyper-Kähler manifolds  $X_k$  is in general given by a sum of symmetric polynomials of  $\left\{\left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^2, \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^2, \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^2\right\}$  with order  $k$ . In the case of a polynomial of the following form,

$$Z_{X_k}^{(1)}(z; \tau) = c_k 2^{2k} \left[ \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^{2k} + \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^{2k} + \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^{2k} \right], \tag{4.47}$$

we can use the character decomposition of the completely degenerate configuration (4.44)–(4.46). Here we introduced a factor  $2^{2k}$  for convenience and the parameter  $c_k \in \mathbb{Z}_{>0}$  is to be fixed later so that  $Z_{X_k}^{(1)}$  contains the identity representation with multiplicity 1 in the NS sector.

In the case of a polynomial of the form

$$\left[ \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^{2k_2} \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^{2k_3} \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^{2k_4} + \text{permutations} \right],$$

one can use the character decomposition (4.35) corresponding to the choice of parameters  $w_1, \dots, w_{P-1}$ :

$$\mathbf{w}_{k_2, k_3, k_4} = \left\{ \begin{array}{l} w_1, \dots, w_k \quad \# \left(w_a = \frac{1}{2}\right) = k_2, \# \left(w_a = \frac{1+\tau}{2}\right) = k_3, \# \left(w_a = \frac{\tau}{2}\right) = k_4, \\ k_2 + k_3 + k_4 = k. \end{array} \right\}.$$

In fact, the Jacobi form corresponding to the vector  $\mathbf{w}_{k_2, k_3, k_4}$  is given by

$$J_{k+1}(z; \mathbf{w}_{k_2, k_3, k_4}; \tau) = \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^{2k_2} \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^{2k_3} \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^{2k_4}.$$

Let us first consider the completely degenerate case. Using (4.44)–(4.46) and (4.25), we obtain

$$c_k^{-1} 2^{-2k} Z_{X_k}^{(1)}(z; \tau) = 3 \text{ch}_{k, h=\frac{k}{4}, \ell=0}^{\tilde{R}}(z; \tau) - \sum_{a=1}^k \sum_{w \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}} H_{k+1}^{(a)}(w; \tau) \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{k-1, \frac{a-1}{2}}(z; \tau).$$

It turns out that the coefficients of lowest powers of the Fourier expansion of  $H_P^{(a)}(w; \tau)$  with  $\tau \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}$  are conveniently computed from the expansion of Jacobi forms (4.43). Namely, by use of (4.27) and (A.2) with the help of an identity of the Chebyshev polynomial,

$$(e^{\pi iz} - e^{-\pi iz})^{2n} = (-1)^n \sum_{k=0}^n (-1)^k \frac{2(k+1)}{n+k+2} \binom{2n+1}{n-k} I_{\frac{k}{2}}(z),$$

we find that

$$\begin{aligned}
 H_P^{(a)}\left(\frac{1}{2}; \tau\right) &= q^{-\frac{a^2}{4P}}\left(\alpha_P^{(a)} + \mathcal{O}(q)\right), \\
 H_P^{(a)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{-\frac{a^2}{4P}+\frac{a}{2}}\left(\beta_{P-1, P-1-a} + \mathcal{O}(q^{\frac{1}{2}})\right), \\
 H_P^{(a)}\left(\frac{\tau}{2}; \tau\right) &= q^{-\frac{a^2}{4P}+\frac{a}{2}}\left((-1)^a \cdot \beta_{P-1, P-1-a} + \mathcal{O}(q^{\frac{1}{2}})\right).
 \end{aligned}
 \tag{4.48}$$

Here we have defined  $\alpha_P^{(a)}$  ( $a = 1, 2, \dots, P - 1$ ) and  $\beta_{n,k}$  by

$$\begin{aligned}
 \alpha_P^{(a)} &= \sum_{m=a}^{P-1} \frac{(-1)^{m+a}}{2^{2m-1}} \frac{a}{m+a} \binom{P-1}{m} \binom{2m-1}{m-a}, \\
 \beta_{n,k} &= \frac{2(k+1)}{n+k+2} \binom{2n+1}{n-k}.
 \end{aligned}
 \tag{4.49}$$

These results agree with Fourier coefficients of  $H_P^{(a)}(z; \tau)$  computed directly from (4.28) in appendix B.

As we see from the appendix, the  $q$ -expansion of  $H_P^{(a)}\left(\frac{1}{2}; \tau\right)$  is always with integer powers of  $q$  (up to an overall factor  $q^{-\frac{a^2}{4P}}$ ) while the sum of  $H_P^{(a)}\left(\frac{1+\tau}{2}; \tau\right)$  and  $H_P^{(a)}\left(\frac{\tau}{2}; \tau\right)$  has either all integer powers (for  $a = \text{even}$ ) or half-integer powers (for  $a = \text{odd}$ ) up to an overall factor  $q^{-\frac{a^2}{4P}+\frac{a}{2}}$ . The sum of  $H_P^{(a)}$ -functions have the  $q$ -expansion of the form

$$H_P^{(a)}\left(\frac{1}{2}; \tau\right) + H_P^{(a)}\left(\frac{1+\tau}{2}; \tau\right) + H_P^{(a)}\left(\frac{\tau}{2}; \tau\right) = q^{-\frac{a^2}{4(a+1)}} \left[ \alpha_{k+1}^{(a)} - \sum_{n \geq 1} A_{k+1,n}^{(a)} q^n \right].
 \tag{4.50}$$

Substituting above into  $Z_{X_k}^{(1)}$ , we find

$$\begin{aligned}
 c_k^{-1} 2^{-2k} Z_{X_k}^{(1)}(z; \tau) &= 3 \text{ch}_{k, h=\frac{k}{4}, \ell=0}^{\tilde{R}}(z; \tau) \\
 &- \sum_{a=1}^k q^{-\frac{a^2}{4(k+1)}} \left[ \alpha_{k+1}^{(a)} - \sum_{n \geq 1} A_{k+1,n}^{(a)} q^n \right] \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \chi_{k-1, \frac{a-1}{2}}(z; \tau).
 \end{aligned}
 \tag{4.51}$$

Note that while terms with the coefficients  $A_{k+1,n}^{(a)}$  give massive representations with  $h = n+k/4$ , terms with coefficients  $\alpha_{k+1}^{(a)}$  are the sum of massless representations because of relation (4.2).

We then rewrite (4.51) using (4.2) and find

$$\frac{1}{c_k} Z_{X_k}^{(1)}(z; \tau) = \sum_{a=0}^k (-1)^a \gamma_{k,a} \text{ch}_{k, \frac{k}{4}, \frac{a}{2}}^{\tilde{R}}(z; \tau) + \text{massive characters}.
 \tag{4.52}$$

Here a coefficient  $\gamma_{k,a}$  of isospin  $a/2$  massless representation is given by

$$\gamma_{k,a} = 2^{2k} (\alpha_{k+1}^{(a)} - 2\alpha_{k+1}^{(a+1)} + \alpha_{k+1}^{(a+2)}),
 \tag{4.53}$$

where in our convention we have

$$\alpha_{k+1}^{(0)} = 3.$$

As we see from table 1, we have  $\gamma_{k,k} = 1$ , which counts the multiplicity of massless representations with isospin  $\frac{k}{2}$  in the R sector. Since the spectral flow (in the left-moving sector) sends this representation to identity representation in the NS sector while the right-moving sector gives a degeneracy of  $k + 1$  for spin  $k/2$  representation, we should multiply  $Z_{X_k}^{(1)}$



**Table 1.**  $\gamma_{k,a}$  for  $a = 0, 1, 2, \dots, k$ .

$k \backslash a$	0	1	2	3	4	5	6	7	8	9	10
1	10	1									
2	37	4	1								
3	142	14	6	1							
4	554	48	27	8	1						
5	2180	165	110	44	10	1					
6	8621	572	429	208	65	12	1				
7	34 198	2002	1638	910	350	90	14	1			
8	135 934	7072	6188	3808	1700	544	119	16	1		
9	541 084	25 194	23 256	15 504	7752	2907	798	152	18	1	
10	2155 938	90 440	87 210	62 016	33 915	14 364	4655	1120	189	20	1

**Table 2.**  $\frac{k+1}{a+1}\gamma_{k,a}$  for  $a = 0, 1, 2, \dots, k$ .

$k \backslash a$	0	1	2	3	4	5	6	7	8	9	10
1	20	1									
2	111	6	1								
3	568	28	8	1							
4	2770	120	45	10	1						
5	13 080	495	220	66	12	1					
6	60 347	2002	1001	364	91	14	1				
7	273 584	8008	4368	1820	560	120	16	1			
8	1223 406	31 824	18 564	8568	3060	816	153	18	1		
9	5410 840	125 970	77 520	38 760	15 504	4845	1140	190	20	1	
10	23 715 318	497 420	319 770	170 544	74 613	26 334	7315	1540	231	22	1

by an overall factor  $(k + 1)$ : then the multiplicity of identity representation in the NS sector is adjusted to 1. Thus the normalization parameter in (4.47) fixes to

$$c_k = k + 1. \tag{4.54}$$

We see that (4.53) is computed for  $a > 0$  as

$$\begin{aligned} \gamma_{k,a} &= 2^{2k} \sum_{m=a}^k \frac{(-1)^{m+a}}{2^{2m-1}} \frac{a+1}{m+a+2} \binom{k}{m} \binom{2m+1}{m-a} \\ &= \frac{2(a+1)}{k+a+2} \binom{2k+1}{k-a}, \end{aligned} \tag{4.55}$$

and that

$$\gamma_{k,0} = C_{k+1} + 2^{2k+1}, \tag{4.56}$$

with the Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . We then have

$$\frac{k+1}{a+1} \gamma_{k,a} \in \mathbb{Z},$$

which gives the net number of massless representations with spin  $\frac{k-a}{2}$  in the NS sector; see table 2.

We note that a series  $\gamma_{k,a}$  for  $a = 1, 2, \dots, k$  has a combinatorial interpretation [22, lemma 1.18]; we consider a set of sequences of 0's and 1's:

$$V(k) = \{v_1 v_2 \cdots v_{2k+1} | \#(1's) = k + 2, \#(0's) = k - 1\}. \tag{4.57}$$

For each element  $v_1 \cdots v_{2k+1} \in V(k)$ , we have a subsequence  $v_1 v_2 \cdots v_h$  for  $h = 1, 2, \dots, 2k + 1$ . Then we observe that  $\gamma_{k,a}$  denotes a cardinality of a subset  $V(k)$  such that

$$\max_h \{\#(1's) - \#(0's)\} = a + 2.$$

From a viewpoint of statistical mechanics, a set  $V(k)$  of sequences is regarded as a staircase walk which starts from  $(0, 0)$  and ends at  $(k - 1, k + 2)$ . Here 0 (resp. 1) corresponds to a unit right-walk  $\rightarrow$  (resp. up-walk  $\uparrow$ ). Then  $\gamma_{k,a}$  coincides with the number of staircase walks which touch a line  $y = x + a + 2$  and do not go beyond the line.

4.7.1. *Other cases.* Along the same strategy as the completely degenerate case, we can decompose symmetric polynomials of the Jacobi theta functions by using a suitable choice of points  $w_1 \cdots, w_{p-1}$ .

Upon symmetrization, we obtain the following:

(a)  $k_2 > k_3 > k_4$ ,

$$\begin{aligned} & 2^{2k_2-1} \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^{2k_2} \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^{2k_3} \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^{2k_4} + \text{other 5 terms} \\ &= \sum_{a=0}^{k_2} (-1)^a (\gamma_{k_2,a} + 2^{2k_2-2k_3} \gamma_{k_3,a} + 2^{2k_2-2k_4} \gamma_{k_4,a}) \text{ch}_{k, \frac{k}{4}, \frac{a}{2}}^{\tilde{R}}(z; \tau) + \text{massive characters}, \end{aligned} \tag{4.58a}$$

(b)  $k_2 = k_3 \neq k_4$ ,

$$\begin{aligned} & 2^{2k_2-1} \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^{2k_2} \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^{2k_4} + \text{other 2 terms} \\ &= \sum_{a=0}^{\max(k_2, k_4)} (-1)^a (\gamma_{k_2,a} + 2^{2k_2-2k_4-1} \gamma_{k_4,a}) \text{ch}_{k, \frac{k}{4}, \frac{a}{2}}^{\tilde{R}}(z; \tau) + \text{massive characters}, \end{aligned} \tag{4.58b}$$

(c)  $k_2 = k_3 = k_4$ ,

$$\begin{aligned} & 2^{2k_2} \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^{2k_2} \\ &= \sum_{a=0}^{k_2} (-1)^a \gamma_{k_2,a} \text{ch}_{k, \frac{k}{4}, \frac{a}{2}}^{\tilde{R}}(z; \tau) + \text{massive characters}. \end{aligned} \tag{4.58c}$$

Here  $\gamma_{k,a}$  is defined by (4.53) with

$$\alpha_{k+1}^{(0)} = 1,$$

which coincides with (4.55) for  $a \geq 0$ . Normalization factors in the left-hand sides in (4.58) are fixed so that coefficients of the massless character are integral. We emphasize that the massless isospin  $k/2$  representation appears only in the completely degenerate case  $(k_2, k_3, k_4) = (k, 0, 0)$  for which the overall normalization can be fixed by requiring the multiplicity to be one for the vacuum representation in the NS sector. We cannot fix the overall normalization in the case of other configurations.

4.8. Example: hyper-Kähler manifold with  $\dim_{\mathbb{C}} = 4$

We have two cases,  $(k_2, k_3, k_4) = (2, 0, 0)$  and  $(1, 1, 0)$ . The former case is (4.47) with  $k = 2$ :

$$48 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^4 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^4 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^4 \right] \\ = 111 \operatorname{ch}_{2, \frac{3}{4}, 0}^{\tilde{R}}(z; \tau) - 12 \operatorname{ch}_{2, \frac{3}{4}, \frac{1}{2}}^{\tilde{R}}(z; \tau) + 3 \operatorname{ch}_{2, \frac{3}{4}, 1}^{\tilde{R}}(z; \tau) \\ + (1872q + 26070q^2 + 213456q^3 + 1311420q^4 + \dots) q^{-\frac{1}{12}} B_3^{(1)}(z; \tau) \\ - (510q + 12804q^2 + 126360q^3 + 841176q^4 + \dots) q^{-\frac{1}{3}} B_3^{(2)}(z; \tau).$$

The latter case follows from (4.58b) and reads as

$$2 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \cdot \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 \right] \\ = 4 \operatorname{ch}_{2, \frac{3}{4}, 0}^{\tilde{R}}(z; \tau) - \operatorname{ch}_{2, \frac{3}{4}, \frac{1}{2}}^{\tilde{R}}(z; \tau) + (16q + 55q^2 + 144q^3 + 330q^4 \dots) \\ \times q^{-\frac{1}{12}} B_3^{(1)}(z; \tau) + (10q + 44q^2 + 110q^3 + 280q^4 + \dots) q^{-\frac{1}{3}} B_3^{(2)}(z; \tau).$$

We require that all coefficients of massless and massive characters should be positive integers in the NS sector when we combine the above two cases. Then the elliptic genus of a general hyper-Kähler manifold  $X_2(n)$  with complex dimension 4 may be written as

$$Z_{X_2(n)}(z; \tau) = 48 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^4 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^4 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^4 \right] \\ + 2n \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \cdot \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \cdot \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \cdot \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 \right], \tag{4.59}$$

where  $n$  is an integer satisfying

$$-12 \leq n \leq 51.$$

We have

$$Z_{X_2(n)}(z = 0; \tau) = 6(n + 24), \\ Z_{X_2(n)} \left( z = \frac{1}{2}; \tau \right) = 2(n + 48) + 12288q + 294912q^2 + \dots, \tag{4.60} \\ Z_{X_2(n)} \left( z = \frac{1+\tau}{2}; \tau \right) = 3q^{-1} - 2(n + 12)q^{-\frac{1}{2}} + 828 + \dots.$$

In the mathematical literature, see for instance [19], two representative examples of complex four-dimensional hyper-Kähler manifolds are discussed: the Hilbert scheme of points on  $K3$  surfaces  $K^{[2]}$  and complex tori  $A^{[[3]]}$ . Their topological invariants are given by

$X$	$\chi_X$	$\sigma_X$	$\widehat{A}_X$
$K^{[2]}$	324	156	3
$A^{[[3]]}$	108	84	3.

(4.61)

These values agree exactly with those of (4.60) for  $n = 30$  ( $K^{[2]}$ ) and  $n = -6$  ( $A^{[[3]]}$ ), respectively. We, in particular, predict  $\widehat{A} = 3$  for any hyper-Kähler manifold in four dimensions.

### 5. Concluding remarks

We have developed a method of improving the modular properties of BPS representations in the  $\mathcal{N} = 4$  SCA by taking an analogy with a theory of mock theta functions. We have obtained a systematic treatment of character decomposition which can be applied to study the elliptic genera of hyper-Kähler manifolds. Our treatment of positivity and integrality properties on the infinite series of massive representations is yet incomplete. We would like to discuss these issues in detail and also the case of higher-dimensional hyper-Kähler manifolds in forthcoming publications.

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### Appendix A. Theta functions

#### A.1. Jacobi theta function

The Jacobi theta functions are defined by

$$\begin{aligned} \theta_{11}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})} = -i\theta_1(z; \tau), \\ \theta_{10}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})z} = \theta_2(z; \tau), \\ \theta_{00}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in z} = \theta_3(z; \tau), \\ \theta_{01}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in(z+\frac{1}{2})} = \theta_4(z; \tau), \end{aligned}$$

where we have also shown the relation to the conventional notations. Under the  $S$ -transformation, we have

$$\begin{pmatrix} \theta_{11}(z; \tau) \\ \theta_{10}(z; \tau) \\ \theta_{00}(z; \tau) \\ \theta_{01}(z; \tau) \end{pmatrix} = \sqrt{\frac{i}{\tau}} e^{-\pi i \frac{z^2}{\tau}} \begin{pmatrix} i & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \theta_{11}(\frac{z}{\tau}; -\frac{1}{\tau}) \\ \theta_{10}(\frac{z}{\tau}; -\frac{1}{\tau}) \\ \theta_{00}(\frac{z}{\tau}; -\frac{1}{\tau}) \\ \theta_{01}(\frac{z}{\tau}; -\frac{1}{\tau}) \end{pmatrix}. \tag{A.1}$$

We see that for  $k \in \mathbb{Z}$

$$\begin{aligned} \left(\frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)}\right)^{2k} &= \left(\frac{e^{\pi iz} + e^{-\pi iz}}{2}\right)^{2k} (1 + 2k(e^{\pi iz} - e^{-\pi iz})^2 q + \dots), \\ \left(\frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)}\right)^{2k} &= 1 + 2k(e^{\pi iz} - e^{-\pi iz})^2 q^{\frac{1}{2}} + \dots, \\ \left(\frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)}\right)^{2k} &= 1 - 2k(e^{\pi iz} - e^{-\pi iz})^2 q^{\frac{1}{2}} + \dots. \end{aligned} \tag{A.2}$$

A.2. *Jacobi form*

Jacobi form  $\varphi(z; \tau)$  with weight  $k$  and index  $m$  has the following transformation formulae [11]:

$$\begin{aligned} \varphi\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) &= \tau^k e^{-2\pi i m \frac{z^2}{\tau}} \varphi(z; \tau), \\ \varphi(z; \tau + 1) &= \varphi(z + 1; \tau) = \varphi(z; \tau), \\ \varphi(z + \tau; \tau) &= q^{-m} e^{-4\pi i m z} \varphi(z; \tau). \end{aligned} \tag{A.3}$$

A.3. *Theta functions and characters*

We define the theta function for  $a \bmod 2P$  by (see, e.g., [21])

$$\begin{aligned} \vartheta_{P,a}(z; \tau) &= \vartheta \left[ \begin{matrix} \frac{a}{2P} \\ 0 \end{matrix} \right] (2Pz; 2P\tau) \\ &= \sum_{n \in \mathbb{Z}} q^{\frac{(2Pn+a)^2}{4P}} e^{2\pi i z(2Pn+a)}. \end{aligned} \tag{A.4}$$

A set of functions  $\{\vartheta_{P,a}(z; \tau) | a \bmod 2P\}$  spans a  $(2P)$ -dimensional space of entire function  $f(z)$  such that

$$f(z + 1) = f(z), \quad f(z + \tau) = e^{-4\pi i P z - 2\pi i P \tau} f(z).$$

The  $S$ -transformation formula is

$$\vartheta_{P,a}(z; \tau) = \sqrt{\frac{i}{\tau}} \frac{1}{\sqrt{2P}} e^{-\pi i \frac{2P}{\tau} z^2} \sum_{b=0}^{2P-1} e^{\frac{ab}{P} \pi i} \vartheta_{P,b}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right). \tag{A.5}$$

With the help of this identity, we have

$$(\vartheta_{P,a} - \vartheta_{P,-a})(z; \tau) = i \sqrt{\frac{i}{\tau}} e^{-2\pi i \frac{P}{\tau} z^2} \sum_{b=1}^{P-1} \mathbf{S}(P)_{ab} (\vartheta_{P,b} - \vartheta_{P,-b})\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \tag{A.6}$$

where

$$\mathbf{S}(P)_{ab} = \sqrt{\frac{2}{P}} \sin\left(\frac{ab}{P} \pi\right). \tag{A.7}$$

The character of the level- $k$   $SU(2)$  affine algebra is given by [15]

$$\chi_{k,\ell}(z; \tau) = \frac{\vartheta_{k+2,2\ell+1} - \vartheta_{k+2,-2\ell-1}}{\vartheta_{2,1} - \vartheta_{2,-1}}(z; \tau). \tag{A.8}$$

Note that the Macdonald–Weyl denominator identity proves

$$(\vartheta_{2,1} - \vartheta_{2,-1})(z; \tau) = -i\theta_{11}(2z; \tau).$$

We define the weight-3/2 modular form from the  $SU(2)$  affine character as

$$\frac{\Psi_P^{(a)}(\tau)}{[\eta(\tau)]^3} = \chi_{P-2, \frac{a-1}{2}}(0; \tau) = \frac{\vartheta_{P,a} - \vartheta_{P,-a}}{\vartheta_{2,1} - \vartheta_{2,-1}}(0; \tau), \tag{A.9}$$

where  $a \in \mathbb{Z}$  satisfying  $0 < a < P$ . We see that the modular form has a Fourier transformation as

$$\Psi_P^{(a)}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \psi_{2P}^{(a)}(n) q^{\frac{n^2}{4P}}, \tag{A.10}$$

where we have used an odd periodic function

$$\psi_{2P}^{(a)}(n) = \begin{cases} \pm 1, & n = \pm a \pmod{2P}, \\ 0, & \text{otherwise.} \end{cases}$$

See that, in the case  $P = 2$ , we have

$$\Psi_2^{(1)}(\tau) = [\eta(\tau)]^3.$$

The modular transformation properties are summarized as

$$\begin{aligned} \Psi_P^{(a)}(\tau) &= \left(\frac{i}{\tau}\right)^{3/2} \sum_{b=1}^{P-1} \mathbf{S}(P)_{ab} \Psi_P^{(b)}\left(-\frac{1}{\tau}\right), \\ \Psi_P^{(a)}(\tau + 1) &= e^{\frac{a^2}{2P}\pi i} \Psi_P^{(a)}(\tau). \end{aligned} \tag{A.11}$$

### Appendix B. Fourier expansions of the harmonic Maass forms

We give the Fourier expansion of the harmonic Maass forms  $M_P^{(a)}(z; \tau)$  (4.28) at  $z \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}$ , which are useful in the character decomposition of the elliptic genera in terms of the level  $k = P - 1$  SCFT. We have directly computed (4.28) using Mathematica and Maple:

- $P = 2$

$$\begin{aligned} \mu\left(\frac{1}{2}; \tau\right) &= h_2(\tau) \cdot \eta(\tau) = \frac{q^{-\frac{1}{8}}}{4} [1 + 3q - 7q^2 + 14q^3 - 21q^4 + \dots] \\ \mu\left(\frac{1+\tau}{2}; \tau\right) &= h_3(\tau) \cdot \eta(\tau) = q^{\frac{3}{8}} [2 - 6q^{\frac{1}{2}} + 14q - 28q^{\frac{3}{2}} + 54q^2 - 98q^{\frac{5}{2}} + 168q^3 - \dots] \\ \mu\left(\frac{\tau}{2}; \tau\right) &= h_4(\tau) \cdot \eta(\tau) = q^{\frac{3}{8}} [-2 - 6q^{\frac{1}{2}} - 14q - 28q^{\frac{3}{2}} - 54q^2 - 98q^{\frac{5}{2}} - 168q^3 - \dots], \end{aligned} \tag{B.1}$$

- $P = 3$

$$\begin{aligned} H_3^{(1)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{1}{8}}}{8} [3 + 8q - 25q^2 + 72q^3 - \dots] \\ H_3^{(1)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{5}{12}} [4 - 20q^{\frac{1}{2}} + 80q - 270q^{\frac{3}{2}} + 812q^2 - 2228q^{\frac{5}{2}} + 5680q^3 - 13650q^{\frac{7}{2}} + \dots] \\ H_3^{(1)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{5}{12}} [-4 - 20q^{\frac{1}{2}} - 80q - 270q^{\frac{3}{2}} - 812q^2 - 2228q^{\frac{5}{2}} - 5680q^3 - 13650q^{\frac{7}{2}} - \dots] \\ H_3^{(2)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{1}{3}}}{16} [1 + 10q - 20q^2 + 40q^3 + \dots] \\ H_3^{(2)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{2}{3}} [5 - 32q^{\frac{1}{2}} + 134q - 448q^{\frac{3}{2}} + 1315q^2 - 3520q^{\frac{5}{2}} + 8764q^3 - 20608q^{\frac{7}{2}} + \dots] \\ H_3^{(2)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{2}{3}} [5 + 32q^{\frac{1}{2}} + 134q + 448q^{\frac{3}{2}} + 1315q^2 + 3520q^{\frac{5}{2}} + 8764q^3 + 20608q^{\frac{7}{2}} + \dots], \end{aligned} \tag{B.2}$$

- $P = 4$

$$\begin{aligned} H_4^{(1)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{1}{16}}}{64} [29 + 70q - 258q^2 + 911q^3 + \dots] \\ H_4^{(1)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{7}{16}} [6 - 42q^{\frac{1}{2}} + 238q - 1134q^{\frac{3}{2}} + 4718q^2 + 59754q^{\frac{5}{2}} - 188480q^3 + \dots] \\ H_4^{(1)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{7}{16}} [-6 - 42q^{\frac{1}{2}} - 238q - 1134q^{\frac{3}{2}} - 4718q^2 - 59754q^{\frac{5}{2}} - 188480q^3 - \dots] \end{aligned} \tag{B.4}$$

$$\begin{aligned}
 H_4^{(2)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{1}{8}}}{8} [1 + 8q - 21q^2 + 56q^3 + \dots] \\
 H_4^{(2)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{3}{8}} [14 - 128q^{\frac{1}{2}} + 762q - 3584q^{\frac{3}{2}} + 14434q^2 + 51840q^{\frac{5}{2}} - 170212q^3 + \dots] \\
 H_4^{(2)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{3}{8}} [14 + 128q^{\frac{1}{2}} + 762q + 3584q^{\frac{3}{2}} + 14434q^2 + 51840q^{\frac{5}{2}} + 170212q^3 + \dots]
 \end{aligned}
 \tag{B.5}$$

$$\begin{aligned}
 H_4^{(3)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{9}{16}}}{64} [1 + 21q - 7q^2 - 77q^3 + \dots] \\
 H_4^{(3)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{15}{16}} [14 - 140q^{\frac{1}{2}} + 852q - 3990q^{\frac{3}{2}} + 15836q^2 + 55890q^{\frac{5}{2}} - 180298q^3 + \dots] \\
 H_4^{(3)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{15}{16}} [-14 - 140q^{\frac{1}{2}} - 852q - 3990q^{\frac{3}{2}} - 15836q^2 - 55890q^{\frac{5}{2}} - 180298q^3 - \dots].
 \end{aligned}
 \tag{B.6}$$

•  $P = 5$

$$\begin{aligned}
 H_5^{(1)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{1}{20}}}{128} [65 + 144q - 591q^2 + \dots] \\
 H_5^{(1)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{9}{20}} [8 - 72q^{\frac{1}{2}} + 528q - 3252q^{\frac{3}{2}} + 16560q^2 - 71268q^{\frac{5}{2}} + \dots] \\
 H_5^{(1)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{9}{20}} [-8 - 72q^{\frac{1}{2}} - 528q - 3252q^{\frac{3}{2}} - 16560q^2 - 71268q^{\frac{5}{2}} - \dots]
 \end{aligned}
 \tag{B.7}$$

$$\begin{aligned}
 H_5^{(2)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{1}{256}}}{256} [46 + 315q - 980q^2 + \dots] \\
 H_5^{(2)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{4}{5}} [27 - 320q^{\frac{1}{2}} + 2468q - 14976q^{\frac{3}{2}} + 77022q^2 - 349248q^{\frac{5}{2}} + \dots] \\
 H_5^{(2)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{4}{5}} [27 + 320q^{\frac{1}{2}} + 2468q + 14976q^{\frac{3}{2}} + 77022q^2 + 349248q^{\frac{5}{2}} + \dots]
 \end{aligned}
 \tag{B.8}$$

$$\begin{aligned}
 H_5^{(3)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{9}{128}}}{128} [5 + 80q - 90q^2 - \dots] \\
 H_5^{(3)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{21}{20}} [48 - 630q^{\frac{1}{2}} + 4984q - 30072q^{\frac{3}{2}} + 151776q^2 - 669194q^{\frac{5}{2}} + \dots] \\
 H_5^{(3)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{21}{20}} [-48 - 630q^{\frac{1}{2}} - 4984q - 30072q^{\frac{3}{2}} - 151776q^2 - 669194q^{\frac{5}{2}} - \dots]
 \end{aligned}
 \tag{B.9}$$

$$\begin{aligned}
 H_5^{(4)}\left(\frac{1}{2}; \tau\right) &= \frac{q^{-\frac{4}{256}}}{256} [1 + 36q + 96q^2 + \dots] \\
 H_5^{(4)}\left(\frac{1+\tau}{2}; \tau\right) &= q^{\frac{6}{5}} [42 - 576q^{\frac{1}{2}} + 4614q - 27776q^{\frac{3}{2}} + 138567q^2 - 597032q^{\frac{5}{2}} + \dots] \\
 H_5^{(4)}\left(\frac{\tau}{2}; \tau\right) &= q^{\frac{6}{5}} [42 + 576q^{\frac{1}{2}} + 4614q + 27776q^{\frac{3}{2}} + 138567q^2 + 597032q^{\frac{5}{2}} + \dots].
 \end{aligned}
 \tag{B.10}$$

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